Math 246A Lecture 18 Notes

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1 Runge's Theorem and Cauchy's Theorem for 1-Forms

1.1 Runge's Theorem

Theorem 1.1 (Runge)**.** *Let Ω be a domain, $f \in H(\Omega)$, and $K \subseteq \Omega$ be compact. Then there exist rational functions $R_n(z)$ with poles in $\mathbb{C}^* \setminus \Omega$ such that*

$$\sup_K |f(z) - R_n(z)| \to 0.$$

Proof. Let $0 < \alpha < \operatorname{dist}(K, \Gamma)$, and partition $\mathbb{C}$ into squares S_j of side length $\delta < \alpha/\sqrt{2}$ with sides parallel to the axes. Let $\hat{K} = \hat{K}_\Omega = K \cup \{U \subseteq \Omega : U \text{ component in } \mathbb{C}^* \setminus K\}$. Then

$$\sup_{\hat{K}} |q(z)| = \sup_K |q(z)|$$

for all $q \in H(\Omega)$. Replace K by $\hat{K} \supseteq K$. Then $S_j \cap \hat{K} = \varnothing \implies s_j \subseteq \Omega$. Let $\Gamma = \sum_{s_j \cap \hat{K} = \varnothing} \partial S_j$ where the S_j are oriented counterclockwise, and we cancel opposite sides. Then $\hat{K} \cap \Gamma = \varnothing$, $\Gamma \subseteq \Omega$, and Γ is piecewise C^1. By Cauchy's theorem for rectangles, if $z \in \hat{K}$, then

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

1. Step 1: There exist $\zeta_1, \dots \zeta_n \in \Gamma$ and $c_1, \dots, c_n \in \mathbb{C}$ such that

   $$\sup_{\hat{K}} |f(z) - \sum_{j=1}^n \frac{c_j}{\zeta_j - z}| < \varepsilon.$$

   Let $\eta = \sup_\Gamma |f(z)|$, and let $\eta > 0$ be such that $|\zeta - \zeta'| < \zeta \implies < |f(\zeta) - f(\zeta')| < \alpha\varepsilon$ for $\zeta, \zeta' \in \Gamma$. Also choose $\eta < \alpha^2/M\varepsilon$. Chop up $\Gamma = \sum_{j=1}^n \Gamma_j$, where Γ_j is an arc of length $< \eta$. Then

   $$\sum_{j=1}^n \ell(\Gamma_j) = \ell(\Gamma) < \infty.$$

Let $c_j = \frac{1}{2\pi i} f(\zeta_j) \int_{\Gamma_j} d\zeta$. Then

$$
\begin{aligned}
\left| f(z) - \sum_{j=1}^{n} \frac{c_j}{\zeta_j - z} \right| &= \left| \sum_{j=1}^{n} \frac{1}{2\pi} \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} - \frac{f(\zeta_j)}{\zeta_j - z} \, d\zeta \right| \\
&\leq \frac{1}{2\pi} \sum_{j=1}^{n} \int \frac{|f(\zeta) - f(\zeta_j)|}{|\zeta - z|} \, ds + \frac{1}{2\pi} \sum_{j=1}^{n} |f(\zeta_j)| \int_{\Gamma_j} \left| \frac{1}{\zeta - z} - \frac{1}{\zeta_j - z} \right| \\
&= A + B,
\end{aligned}
$$

where $A \leq 1/(2\pi)\varepsilon\alpha\ell(\Gamma)$, and $B \leq 1/(2\pi)Mn/\alpha^2\ell(\Gamma) < 1/(2\pi)\varepsilon\ell(\Gamma)$.

2. Step 2: Let $\zeta_j \notin \hat{K}$. Then U is the component of $\mathbb{C} \setminus K$ wihth $\zeta_j \in U$ and $a_j \in U \setminus \Omega$. We can accomplish this step and finish the proof by the following lemma. □

Lemma 1.1. *For all $\zeta_j \in U$, there exists a sequence P_n of polynomials so that*

$$P_n(1/(z - a_j)) \to \frac{1}{z - \zeta_j}$$

uniformly on $\hat{K}$.

Proof. Let $V = \{b \in U : \text{ claim is true}\}$. Then $a \in V$. V is closed in U. But also, V is open; if $b \in V$, then let $B = |\zeta - b| < \operatorname{dist}(b, \hat{K})/2$. Then

$$\frac{1}{z - \zeta} = \frac{z}{(z - b)0(\zeta - b)} = \frac{1}{z - b} \sum_{n=0}^{\infty} \left(\frac{\zeta - b}{z - b} \right)^n$$

for all $\zeta \in V$. By the connectedness of U, $V = U$. □

Corollary 1.1 (Runge's theorem for polynomials)**.** *Let $K \subseteq \mathbb{C}$ be compact be such that $\mathbb{C}^* \setminus K$ is connected with $f \in H(\Omega)$. Then there exist a sequence of polynomials $P_n(z)$ such that $P_n(z) \to f(z)$ uniformly on K.*

1.2 Cauchy's theorem for 1-forms

Theorem 1.2 (Cauchy's theorem for 1-forms)**.** *Let Ω be a domain, and let $\gamma \subseteq \Omega$ be a cycle homologous to 0. Let $P\,dx + Q\,dy$ be a closed C^2 1-form ($P_y = Q_x$). Then*

$$\int_\gamma P\,dx + Q\,dy.$$

Lemma 1.2. *There exists a cycle σ such that*

1. *σ consists of horizontal and vertical segments*

2. *for all closed* $P\,dx + Q\,dy$ *on* Ω

$$\int_\gamma P\,dz + Q\,dy = \int_\sigma P\,dx + Q\,dy.$$

3. $\sigma \sim 0$.

Proof. Let $\alpha = \operatorname{dist}(\gamma, \partial\Omega) > 0$. Let $\gamma = \gamma_1 + \cdots + \gamma_m$, where $\gamma_j : [a_j, b_j] \to \Omega$ and $a_{j+1} > b_j$. So $\gamma(t) = \sum_j \gamma - j(t)\mathbb{1}_{[a_j,b_j]}(t)$. There exists $\delta > 0$ such that $|t-s| < \delta|$ such that $|\gamma(t) - \gamma(s)| < \alpha/\sqrt{2}$. Now let $a_j \leq t_k < t_{k+1} \leq b_j$. Let $B)k = \{z : |z - \gamma(t_k)| < \alpha\} \subseteq \Omega$. Let $\sigma_k^{(j)}$ be the polygonal path connecting $\gamma(t_k)$ and $\gamma(t_{k+1})$. Then let $\sigma = \sum_j \sum_k \sigma_k^{(j)}$.

Then

$$\int_{\gamma_k^{(j)}} P\,dz + Q\,dy = \int_{\sigma_k^{(j)}} P\,dx + Q\,dy$$

for each j, k. So it now suffices to show that $\int_\gamma P\,dz + Q\,dy = 0$. Extend all lines containing segments of σ to full lines. These bound certain bounded rectangles $R_1, \ldots, R_n$ and bunbounded rectangles $R'_1, \ldots, R'_n$. We define $\tilde{\sigma} = \sum n(\sigma, a_j)\partial R_j$. Next time, we will show that $\tilde{\sigma} = \sigma$ and that the integral over $\tilde{\sigma}$ is zero. $\square$